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ABSTRACT

The Poisson probability distribution is seen to provide a mathematical model from which useful information can be obtained in practical applications. The distribution and some situations to which it applies are studied, and ways to find answers to practical questions are noted. The unit includes exercises and a model exam, and provides answers to both. Appendices cover basic probability concepts and derivation of Erlang's formula. (MP)

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UNIT 340

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

THE POISSON RANDOM PROCESS

by Carroll O. Wilde

The Poisson probability distribution provides a mathematical model from which we can obtain useful information in practical applications. In this unit we study the Poisson distribution and some situations to which it applies, and we show how to find answers to practical questions that arise in these contexts.

APPLICATIONS OF PROBABILITY THEORY TO OPERATIONS RESEARCH

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THE POISSON RANDOM PROCESS

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Title: THE POISSON RANDOM PROCESS

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Prerequisite Skills:
1. A completed or concurrent course in introductory probability.
2. The ability to use summation notation.
3. Know the basic concepts of derivative and integral from calculus.

Output Skills:
Be able to obtain practical information about:
1) random arrival patterns
2) inter-arrival times, or "gaps" between arrivals
3) waiting line buildup
4) service loss rates

from the Poisson distribution, the exponential distribution, and Erlang's formulas.

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MODULES AND MONOGRAPHS IN UNDERGRADUATE

MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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I. INTRODUCTION

The Poisson probability distribution provides a mathematical model from which we can obtain useful information in practical applications. In this unit we study the Poisson distribution and some situations to which it applies, and we show how to find answers to practical questions that arise in these contexts.

We begin by posing two problems that are related to the Poisson distribution. Although imprecisely formulated here, these problems do illustrate some underlying ideas.

Problem A. Suppose that you live in an isolated community where fires break out at random at an average of 3 per day. If fires require an average of 1 hour to fight, how many firefighting units should your fire station have to make the community "safe?"

Problem B. Suppose that you own a hardware store that carries brooms. Your merchandise is restocked only at the close of your business week, each Saturday afternoon. You have limited storage space and therefore wish to keep inventory levels at a minimum. If customers who buy brooms arrive at random times and at an average rate of 10 per week, how many brooms should you have on hand each Monday morning?

Problems A and B illustrate practical situations that involve "random arrivals" at given locations. Other examples are customer arrivals at a barbershop, at a supermarket (for goods), at a supermarket checkout counter (for service), or at a gasoline station. Further examples are the birth of a member of a population, messages arriving at a message center, letter drafts arriving in a typist's in-basket, alpha particles stimulating a Geiger counter, the commission of felonies in a police precinct, and manuscripts arriving at UMAP!

Corresponding to random arrivals we also have

"random departures". There are customer departures from a barbershop after a haircut, messages disseminated after processing, deaths of members of a population, typed letters departing from a typewriter, particle emissions from a radioactive source, and light bulbs burning out.

Our study is focused on systems in which goods or services are provided to customers who arrive at random times. We consider two basic types of systems, which correspond to Problems A and B.

A'. *Service-Oriented Systems.* Here we are concerned with the impact of arrival randomness on the requirements for *speed of service* necessary to meet customer expectations and demands for promptness. In addition to the fire protection example, speedy service is important for barbershops, banks, clerical stations, message centers, computer centers, supermarkets, and factory assembly lines.

B'. *Goods-Oriented Systems.* Here we consider the impact of arrival randomness on requirements for *inventory* necessary to meet customer expectation for goods and supplies. The problem is important in such retail establishments as hardware stores, supermarkets and gas stations. It is also important, albeit to a lesser degree, for service agencies such as barber shops and television repair centers, since service often involves supplies or parts of some kind.

In many practical situations, the randomness of arrivals and departures can be described by a Poisson random process (see Section 2 for use of this terminology.) In planning effective goods- or service-oriented systems, we use this process to study arrival rates, "gaps" between arrivals, (or waiting times), service (or "holding") times, and the effects of different waiting line patterns. (Is there one waiting line leading to many servers? Are there parallel waiting lines, each with its own server?

Is there an arrangement for priority interruptions?)

The study of "gaps" is important in its own right, without regard to service- or goods-oriented systems as we have described above. For example, traffic control involves times and distances between vehicles arriving at an intersection, and a typesetter is concerned with the number of error-free pages between typos!

In Sections 2 and 3 we briefly review some fundamental of the Poisson distribution and the closely related exponential distribution. You may wish to turn directly to Section 4, where we present some of Erlang's formulas that are used in the applications. An appendix (Appendix A) is included for those who want a brief review of some of the fundamentals of probability theory.

2. THE POISSON DISTRIBUTION

The "Poisson distribution" is actually a whole family of probability distributions, one for each value of a parameter $\alpha > 0$. The outcomes are 0, 1, 2, ..., and the probability function is given by

$$(2.1) \qquad P(n) = e^{-\alpha}\alpha^n/n!, \qquad n = 0, 1, 2, \ldots .$$

The mean, μ, of the Poisson distribution with parameter α turns out to be the parameter itself:

$$(2.2) \qquad \mu = \alpha.$$

The Poisson distribution is appropriate in the following type of situation. Suppose that in any interval of length t (which often represents time, but the interpretation is valid also for distance), we may have any number of occurrences of a particular phenomenon at random points of the interval. We called these occurrences "arrivals" or "departures" in the examples of the previous section. Denote by $P_n(t)$ the probability that there will be exactly n such occurrences in the interval. Then

under the hypothesis indicated below, the Poisson distribution is given by

$$(2.3) \qquad P_n(t) = e^{-at}(at)^n/n!, \quad n = 0, 1, 2, \ldots,$$

where a is a positive constant whose meaning we shall explain presently. When we interpret the Poisson distribution (2.1) in this way as a function of time t, we often refer to it as the Poisson random (stochastic) process.

Equation (2.3) is based on several hypotheses, which may be stated roughly as follows.

(a) The function $P_n(t)$ is well defined, in the sense that only the length of the interval matters and not its location.

(b) Any two nonoverlapping intervals, when regarded as events, are statistically independent.

(c) For small intervals of length Δt we have

(i) $P_1(\Delta t) \approx a\Delta t$;

(ii) $\sum_{n=2}^{\infty} P_n(\Delta t) \approx 0.$

The discussion in Appendix B should help to clarify (c). Assumption c(ii) means that when an interval is very small, the probability of more than one arrival in the interval is negligible.

The mean $\mu_a(t)$ of the Poisson process determined by Equation (2.3) is given by

$$(2.4) \qquad \mu_a(t) = at.$$

This result shows the meaning of a, the proportionality constant from c(i): taking t = 1 we see that a *is the mean number of arrivals or departures per unit of time or length.*

We observe that the notation $\mu_a(t)$ indicates the dependence of the mean on both a and t, and that t is

usually regarded as a variable and a as a parameter. We often abbreviate to $\mu(t)$ when a is understood, and to simply μ when a and t are both understood.

Exercise 1. Suppose that customer arrival times at a barbershop have a Poisson distribution with mean arrival rate of one customer every 10 minutes. This is a mean arrival rate of 6 customers per hour, so that with t in hours, $a = 6$.

a. Find the mean number of arrivals in any half-hour period.

b. Find the probability that exactly 0, 1 or 2 customers will arrive in a given half-hour period.

---

3. INTER-ARRIVAL GAPS

Suppose we have a Poisson arrival process with mean rate a. To find the probability distribution for the length of time or distance between any two consecutive arrivals, we first consider the length to the first arrival, starting from zero. Let f denote the probabili density function of this distribution, t any positive number, and A the interval from 0 to t. Since we agreed to start at zero, we have $f(u) = 0$ for all $u < 0$. Hence,

$$P(A) = \int_0^t f(u)du$$

$$= \int_{-\infty}^t f(u)du$$

$$= F(t)$$

where F is the cumulative distribution function. But the probability that the first arrival occurs in the interval from 0 to t must also be given by the sum

$$\sum_{n=1}^{\infty} P_n(t),$$

because this sum represents the probability that there will be one or more arrivals in the interval. Therefore

$$F(t) = \sum_{n=1}^{\infty} P_n(t) = 1 - P_0(t) = 1 - e^{-at}$$

from which we obtain (by differentiating)

$$(3.1) \quad f(t) = \begin{cases} ae^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

Since the arrival distribution is independent of interval location, Equation (3.1) also provides the probabilities for the "gaps" between any consecutive arrivals. That is, the probability that the waiting time between any two consecutive arrivals will be anywhere from t_1 to t_2 minutes long will be

$$\int_{t_1}^{t_2} ae^{-at} dt.$$

Thus, this exponential distribution represents the inter-arrival, or gap, distribution.

The mean M of the distribution determined by Equation (3.1) is given by

$$(3.2) \quad M = 1/a.$$

This result agrees with our interpretation of a as the mean arrival rate. For example, an average arrival rate of 6 customers per hour corresponds to an average wait of 1/6 of an hour, or 10 minutes, between arrivals.

Exercise 2. In the barbershop problem of Exercise 1, suppose that a customer arrives at exactly 2 P.M. Find the probability that the next customer will arrive by 2:30.

---

4. ERLANG'S LOSS FORMULA

In this section we present formulas that are useful in planning effective customer service. One of these formulas is often called *Erlang's loss formula*, and the others are closely related to it. Derivations are

indicated in Appendix B.

The context of the problem is a service facility with c servers, at which customers arrive at random times and at mean rate a. This assumption on customer arrivals should be quite reasonable for anyone who is comfortable with the ideas outlined in Section 2.

Departure times, on the other hand, are not so easily described, because they depend on two factors: (1) the arrival times, and (2) the length of time required for service. In real applications there are many possibilities for service (or holding) time distributions. Some common ones are exponential (think about it -- just like inter-arrival times!!), uniform over an interval, and even deterministic, i.e. constant. What is important for us is that departure times satisfy the same basic hypotheses (those that led to the Poisson distribution in the first place) as the arrivals, except that the "service rate" is defined in terms of conditional probability. Roughly, b is the average number of customers that a consistently busy server will serve per unit time. The basic condition is that for any time t and any very small Δt we have

$$P(\text{departure in } [t,t+\Delta t] \mid \text{server busy at } t) \approx b\Delta t.$$

We assume that each of the c servers provides service at the same rate b, and we consider only the case $cb > a$. (If $cb \leq a$, then the number of unserviced customers would increase without bound.)

Let K be the "system size", i.e. the number of servers plus the length of the waiting line, or queue, that is allowed to form. The waiting line capacity, then, is $K - c$. We use the notation $K = \infty$ to indicate that no a priori limitation is placed on the number of customers who may be in line for service.

Denote by $p_n(t)$ the probability that there are

exactly n customers in the system at time t, where $t \geq 0$. The number of customers, n, may be any integer from 0 to K, or any nonnegative integer when $K = \infty$. In many practical situations the probability distribution of $p_n(t)$ settles down to a steady-state distribution that is independent of t (regardless of the number of customers in the system at time $t = 0$; it may take a while to reach this situation). Thus, in the steady-state, $p_n(t)$ is constant with respect to t, and we denote it by simply p_n. We present the probability distribution of p_n, distinguishing three cases.

Case I. $K = c$. In this case, no line is allowed to form. With $c = 1$, this case may provide a model for incoming calls to an ordinary telephone: when the line is busy, the incoming call is not put "in line" for subsequent answering; the call is "lost" (unless the caller tries again later, and this could affect the results, as we shall see in the examples). The case $K = c$ also provides information on the fire protection problem, since most fires spread very fast and hence (for an approximation), it may be reasonable to regard calls that are not answered immediately as losses.

The steady-state distribution for $K = c$ is known as *Erlang's first formula* and is given by

$$(4.1) \qquad P_n = \frac{(a/b)^n/n!}{\sum_{i=0}^{c} (a/b)^i/i!} \qquad n = 0, 1, \ldots, c.$$

The ratio a/b is called the *traffic intensity, and we denote it by r.* If a is measured, say, in arrivals per hour, and b in customers served per hour by a consistently busy server, then

$$\frac{a}{b} = \frac{\text{arrivals/hour}}{\text{customers served by a busy server/hour}}.$$

Thus, $r = a/b$ is a measure of the ability of a single server to deal with the traffic stream coming in.

With this notation, Equation (4.1) becomes

$$(4.2) \qquad p_n = \frac{r^n/n!}{\sum_{i=0}^{c} r^i/i!}, \quad n = 0, 1, \ldots, c.$$

Erlang's loss formula is the formula for p_c itself:

$$(4.3) \qquad p_c = \frac{r^c/c!}{\sum_{i=0}^{c} r^i/i!} .$$

This formula yields the probability that the system is full at any time in the steady state. The value of p_c represents the fraction of time that the system is full and hence, since the arrivals occur at random times, p_c must also represent the fraction of customers that are lost, i.e., the probability that an arriving customer will be lost.

For the case $K = c$, the probability that an arriving customer will be lost is

$$(4.4) \qquad p_c = \frac{r^c/c!}{\sum_{i=0}^{c} r^i/i!} , \text{ (where } r = a/b).$$

Case II. $c < K < \infty$. In this case we assume that the number of customers in line for service may not exceed a prescribed finite value $K - c$. This assumption may be appropriate for the barbershop problem, for example. It could be assumed for approximation purposes that there is a fixed number m such that any customer who arrives to find m people in line will not wait.

The probability distribution for p_n in this case is more complicated:

$$(4.5) \qquad p_n = \begin{cases} (r^n/n!)p_0, & n = 0, 1, \ldots, c-1 \\ (r^n/c!c^{n-c})p_0, & n = c, c+1, \ldots, K \end{cases},$$

where

$$(4.6) \qquad p_0 = \frac{1}{\sum_{i=0}^{c-1} r^i/i! + \frac{r^c}{c!} \frac{1-(r/c)^{K-c+1}}{1-r/c}}.$$

While these formulas look forbidding, they are made manageable by the fact that in many practical situations c is relatively small.

Since p_K represents the probability that the system is full, we can calculate that an arriving customer is lost by taking n = K in Equation (4.5):

For the case $c < K < \infty$, the probability that an arriving customer will be lost is

$$(4.7) \qquad p_K = \frac{r^K}{c!c^{K-c}} p_0,$$

where p_0 is given by Equation (4.6).

Case III. $K = \infty$. In this case the queue may contain any number of customers In practice, queues do not become infinite except, perhaps, at some gas stations we've seen recently! However, when K - c is very large we may use the limiting distribution to simplify calculations. The probability distribution for p_n in this case is given by

$$(4.8) \qquad p_n = \begin{cases} (r^n/n!)p_0 & n = 0, 1, \ldots c-1 \\ (r^n/c!c^{n-c})p_0 & n = c, c+1, \ldots \end{cases}$$

where

$$(4.9) \qquad p_0 = \frac{1}{\sum_{i=0}^{c-i} r^i/i! + \frac{r^c}{c!} \frac{1}{1-r/c}}$$

In this case, the total system is never "full", because we can always add an arriving customer to the queue. Thus, in a sense, arriving customers need never be lost, although real customers who arrive to find very long waiting lines may very well decide not to wait for service. Nevertheless, this model is useful in several ways. It provides a good approximation to many real situations, it simplifies calculations because we may use Equation (4.9) instead of (4.6) when K - c is large, and it can be used to find the expected length of the queue and the average total time from customer arrival to departure. We cite one particular result that is of some interest, namely the probability that an arriving customer must wait in line. This event occurs when all c servers are busy, and hence when the total number of customers in the system is at least c. The required probability is therefore $p_c + p_{c+1} + \ldots$, which is a geometric series whose sum is

$$(4.10) \qquad \sum_{n=c}^{\infty} p_n = \frac{(r^c/c!)\,\frac{1}{1-r/c}}{\sum_{i=0}^{c-1} r^i/i! + \frac{r^c}{c!}\,\frac{1}{1-r/s}} .$$

In Section 5 we apply the results of this section in "service-oriented" systems to obtain information for service planning. First, we offer a routine exercise for familiarization with the formulas.

Exercise 3. Suppose we have a service system with c = 3 servers, a waiting line capacity for K - c = 9 customers, an arrival rate of a = 10 customers per hour, and a service rate of b = 5 customers per hour.

a. Calculate the probability p_0 that the system is idle.

b. Find an approximation for p_0 under the assumption K = ∞. Compare your approximation with the result of part (2).

c. Calculate the probability that at least one server is idle.

d. Calculate the probability that an arriving customer must wait for service.

e. Calculate the probability that an arriving customer will be lost.

f. Find an approximation for the probability that an arriving customer will be lost using Equation (4.9) instead of (4.6) to find p_0. Compare your approximation with the result of part (e).

---

5. SERVICE-ORIENTED SYSTEMS

Example 1. *Barbershop.* Suppose that customers arrive at random times during business hours at an average rate of 10 per hour. If we choose a minute as the basic unit of time, then $a = 1/6 \approx .167$ arrivals per minute.

Suppose further that the average haircut takes 10 minutes. Then one barber cannot handle the traffic, but two could. The question is, however, what sort of queueing will occur because of randomness of customer arrivals? For example, we want to know whether the waiting line buildup would at times become so excessive as to drive business away, for then it might be better to staff a third chair.

We use results from Section 4. For this problem there are $c = 2$ servers, $a = 1/6$, $b = 1/10$, and $r = a/b = 5/3$. We assume $K = \infty$ for approximation purposes. Then,

$$p_0 = \frac{1}{1 + 5/3 + (1/2)(5/3)^2 \frac{1}{1-5/6}} = 1/11 \approx .091$$

by Equation (4.9), and from (4.8) we obtain

$$p_1 = (5/3)p_0 = 5/33 \approx .152,$$

$$p_2 = (5/6)p_1 = 25/198 \approx .126,$$

$$p_3 = (5/6)p_2 = 125/1188 \approx .105.$$

In Table 1 we show the probability P_N that there will be a total of N or more customers in the system for a few values of N.

TABLE 1.

The Probability that there are N
or More Customers Waiting or Being Served in
the Barbershop of Example 1, for Selected Values of N

N	P_N
2	.758
3	.631
4	.526
5	.438
6	.365
7	.304
8	.254
9	.211
10	.176

The table shows, for example, with N = 2 that arriving customers will find both barbers busy around 76% of the time, so 3 out of every 4 arriving customers will have at least some wait. With N = 4 we see that more than half the time there will be more than 1 customer waiting, and with N = 8 we see that 1 out of every 4 arrivals will find at least 6 people in line!

We have not tried to find the length of time that customers expect to wait in the queue, but the figures above indicate that with only 2 chairs staffed there is likely to be some excessive waiting. The gain made by adding a third barber is explored in Exercise 4. The ultimate desirability of such a move would depend on factors like labor cost, the increased amount of idle server time, and customer tolerance of waiting lines.

Exercise 4. Suppose that a third barber is added to the shop in Example 1.

a. Find p_0, p_1, p_2, p_3, p_4.

b. Find the probability that all three barbers are busy, hence also the fraction of arriving customers who must wait.

c. Show that more than half the time at least two barbers are idle.

Example 2. *Fire Protection.* Suppose that in a certain community there are three fires per day on the average, so that a may be taken as 1/8, or .125, fire outbreaks per hour. Suppose also that the average time required to fight a fire is one hour, i.e. b = 1. We also assume that no waiting line forms, i.e. K = c, the number of firefighting units, so that fire calls are lost if all c units are busy. We wish to find the smallest value of c for which the community will be "reasonably safe".

We apply Erlang's loss formula, Equation (4.4), with r = 1/8, for several possible values of c.

c = 1: $\quad p_1 = \dfrac{1/8}{1 + 1/8} = 1/9.$

Thus, the probability that a call will be lost is 1/9; that is, on the average every ninth call will be lost. Since there are 3 calls per day, we lose a call every 3 days, which seems to be quite unsafe.

c = 2: $\quad p_2 = \dfrac{(1/8)^2/2}{1 + 1/8 + (1/8)^2/2} = 1/145.$

That is, over the long run we would find that one out of every 145 fire calls would go unanswered; this would occur once every 145/3 = 48-1/3 days, or roughly once every 7 weeks.

Certainly, then, this community must have at least 3 firefighting units -- but then how safe will it be?

c = 3: $\quad p_3 = \dfrac{(1/8)^3/6}{1 + 1/8 + (1/8)^2/2 + (1/8)^3/6} = 1/3,481.$

Therefore, with 3 units there will be one unanswered call every 3,481/3 = 1,160-1/3 days, or around once every 38 months.

With four units the rate drops to one unanswered call every 101 years. (See Exercise 5.) With allowances made for vehicle maintenance and other factors that may affect the accuracy of our results, it looks as if this community could feel reasonably safe with four fire-fighting units.

Exercise 5.

a. Using Erlang's loss formula with c = 4 to find the probability that a fire call will go unanswered when the community of Example 2 has four firefighting units.

b. Convert the result of part (a) to an average rate of years per unanswered calls.

We pause for a moment to point our some of the limitations in mathematical modeling that are illustrated by the fire protection example. We made the simplifying assumption that fires are ignored if they break out when all crews are busy. Is this assumption reasonable? Certainly fires don't queue up and wait to be extinguished! Yet we all know that in real situations a fire crew may be diverted from one fire to another under special circumstances. We have not considered priorities -- we would no doubt sacrifice an isolated building to save a whole section of town. Moreover, our model does not account for seasonal variations, such as higher danger levels in dry seasons and lower in rainy seasons. In addition, we have assumed that the fire station serves an isolated small community; the model is not appropriate for cities where there is cooperation between neighboring fire districts.

This discussion points up the need for care in the interpretation of results. We *can* obtain good insight from mathematical models, but we must remember that approximations involve error, and when errors exceed

tolerable limits we should refine the model. Such model refinements are considered in more advanced studies in operations research.

Example 3. *Telephone Service.* A small business has 3 telephone lines. Calls occur at random times, at an average rate of 40 per 8-hour working day, and the average duration of these calls is 4 minutes. The owner is concerned about the frequency of incoming calls that get a busy signal and the number of potential outgoing calls that are not made because all lines are in use. The owner wants to know if another line should be added. Can you help?

Sure! If we use hours for units, then $a = 40/8 = 5$ calls per hour. In addition, $b = 15$, because a "service rate" of one call in 4 minutes is equivalent to 15 per hour. Thus, traffic intensity is $r = a/b = 1/3$. We are given that the number of servers (lines) is $c = 3$, and if we assume that a potential call is lost when all lines are busy (see the remark below), then $K = c$. By Erlang's loss formula (Equation (4.4)), we have

$$p_3 = \frac{(1/3)^3/3!}{1 + 1/3 + (1/3)^2/2! + (1/3)^3/3!} = \frac{1}{224} \approx .004 .$$

Since there are 40 calls per day, the daily loss rate is about 0.16. Thus, we would expect to lose one call in a 6-day business week.

Remark. The assumption that calls are lost when all lines are busy is highly questionable. Incoming and outgoing calls both may merely be delayed until lines are free, because callers may keep trying to complete their calls. To estimate our error, let's assume the worst, that *no* calls are lost (all callers keep trying when the lines are busy), so that we have the equivalent of a queue with $K = \infty$. The probability that all lines will be occupied is then given by Equation (4.10):

$$\sum_{n=3}^{\infty} p_n = \frac{((1/3)^3/3!)\ \frac{1}{1-1/9}}{1 + 1/3 + (1/3)^2/2! + ((1/3)^3/3!)\frac{1}{1-1/9}}$$

$$= 1/201 \approx .005.$$

Under this assumption, the loss rate would be 0.20 per day, or around one in every 5 business days. We feel that the actual error resulting from the "lost call" assumption should be somewhat less than the error resulting from the worst case assumption of no lost calls.

Exercise 6. Suppose that the owner of the business in Example 3 decided on the basis of our results that two lines would be adequate.

a. Use Erlang's loss formula to calculate the resulting call loss rate (under the assumption that calls are lost when all lines are busy).

b. Find the probability that both lines will be busy using Equation (4.10) (under the assumption that no calls are lost).

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6. GOODS-ORIENTED SYSTEMS

In this section we are concerned with the effects of randomness in customer arrivals on inventory control. The example is rather long, so we present it in two parts.

Example 4. *Inventory Stocking, Part I.* Suppose that customers who buy brooms at a particular store arrive at random times and at an average rate of 10 per week. If brooms can be restocked only on weekends, how many brooms should the merchant have on hand each Monday morning?

If we use weeks as the unit of time, then the mean arrival rate is $a = 10$. Thus, $P_n(t)$ (see Equation (2.3)) represents the probability that n brooms are sold in t weeks. Then for every nonnegative integer n we have $P_n(1) > 0$, hence there is no absolute guarantee that the brooms will be sold out in a week no matter how many are stocked.

Now, because of storage requirements, capital outlay, and problems with deterioration and obsolescence, the merchant wishes to keep the number of brooms on hand each Monday morning reasonably small. At the same time, the demand must be accommodated or sales will be lost. We calculate some probabilities from the Poisson distribution with $a = 10$, $t = 1$:

$P_{5 \text{ or less}}(1) \approx 0.067$ $\quad$ $P_{11 \text{ or less}}(1) \approx 0.697$

$P_{6 \text{ or less}}(1) \approx 0.130$ $\quad$ $P_{12 \text{ or less}}(1) \approx 0.792$

$P_{7 \text{ or less}}(1) \approx 0.220$ $\quad$ $P_{13 \text{ or less}}(1) \approx 0.864$

$P_{8 \text{ or less}}(1) \approx 0.333$ $\quad$ $P_{14 \text{ or less}}(1) \approx 0.917$

$P_{9 \text{ or less}}(1) \approx 0.458$ $\quad$ $P_{15 \text{ or less}}(1) \approx 0.951$

$P_{10 \text{ or less}}(1) \approx 0.583$ $\quad$ $P_{16 \text{ or less}}(1) \approx 0.973.$

Therefore, if 10 brooms are stocked, then the supply will meet the demand in 58 percent of the business weeks. If 15 are stocked, then demand will exceed supply only one week in 20. For 16 brooms, the figure is one week in 36.

Exercise 7. Find the minimum number of brooms that the merchant in Example 4 should stock each week to reduce the sales loss rate to a point where demand exceeds supply only one week every 2 years (or better).

Example 5. *Inventory Stocking, Part II.* In Part I we calculated probabilities that showed the frequency with which the weekly demand will exceed weekly supply for various values of the stocking level. From a practical point of view, we would also want to know the expected number of broom sales lost each week, in order to find the expected dollar loss. In Part II we calculate the expected number of sales lost using the basic definition of mathematical expectation.

To gain some insight, let us suppose that 15 brooms are stocked on Monday morning. If the demand is 15 or less that week, then the number of lost sales is 0. If the demand is 16, the number lost is 1; if 17, the number is 2; and so on. Thus, the possible outcomes are 0, 1, 2, ..., and the associated probabilities are:

$$P(0 \text{ sales lost}) = P_{15 \text{ or less}}(1) \approx 0.9725,$$

$$P(1 \text{ sale lost}) = P_{16}(1) \approx 0.0217$$

$$P(2 \text{ sales lost}) = P_{17}(1) \approx 0.0128,$$

....

Therefore, the expectation of the number of lost sales is

$$\mu = 0(0.9725) + 1(0.0217) + 2(0.0128) + \ldots. \tag{6.1}$$

(see Equation (11.6)' in Appendix A).

We could now calculate the value of μ numerically, adding up enough terms to meet a prescribed accuracy. Such an effort, however, would be clumsy and tedious, and would provide very little insight into the problem. It is much better to derive a general formula for the expected number of sales lost.

Let a be the average demand (so far, we have been taking the value of a as 10), and suppose that the merchant stocks k brooms (k = 15 in our example above). If we let n be the actual demand during a given week, then the number of sales lost is 0 if $n \le k$, 1 for $n = k + 1$, 2 for $n = k + 2$, etc., and by Equation (11.6)' we have

$$\begin{aligned} \mu &= \sum_{n=0}^{k} 0 \cdot P_n(1) + \sum_{n=k+1}^{\infty} (n-k) P_n(1) \\ &= \sum_{n=k+1}^{\infty} n P_n(1) - k \sum_{n=k+1}^{\infty} P_n(1) . \end{aligned} \tag{6.2}$$

Now, since

$$P_n(t) = \frac{e^{-at}(at)^n}{n!} \tag{6.3}$$

from Equation (2.3),

$$\begin{aligned} \sum_{n=1}^{\infty} nP_n(1) &= \sum_{n=1}^{\infty} n \frac{e^{-a}a^n}{n!} \\ &= ae^{-a} \sum_{n=1}^{\infty} \frac{a^{n-1}}{(n-1)!} \qquad (6.4) \\ &= ae^{-a}e^{a} \\ &= a. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(1) &= \sum_{n=0}^{\infty} \frac{e^{-a}a^n}{n!} \\ &= e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} \qquad (6.5) \\ &= e^{-a}e^{a} \\ &= 1. \end{aligned}$$

Accordingly, the formula for μ in (6.2) can be rewritten as

$$\mu = \left[a - \sum_{n=1}^{k} nP_n(1)\right] - k\left[1 - \sum_{n=0}^{k} P_n(1)\right]. \tag{6.6}$$

Equation (6.3) also tells us that

$$\begin{aligned} nP_n(1) &= n \frac{e^{-a}a^n}{n!} \\ &= a \frac{e^{-a}a^{n-1}}{(n-1)!} \qquad (6.7) \\ &= a\, P_{n-1}(1) \end{aligned}$$

When we substitute this last expression for $nP_n(1)$ in Equation (6.6), we obtain

$$\mu = \left[a - \sum_{n=1}^{k} aP_{n-1}(1)\right] - k\left[1 - \sum_{n=1}^{k} P_n(1)\right]$$

$$= a\left[1 - \sum_{n=0}^{k-1} P_n(1)\right] - k\left[1 - \sum_{n=0}^{k} P_n(1)\right]$$

(6.8)

$$= (a - k)\left[1 - \sum_{n=0}^{k} P_n(1)\right] + aP_k(1)$$

$$= aP_k(1) - (k - a)\left[1 - \sum_{n=0}^{k} P_n(1)\right].$$

In words, this last expression for μ says that the average number of sales lost per week is the difference between the following two products:

(1) the mean number of sales per week, times the probability that actual demand will exactly equal the supply;

(2) the excess of inventory stocked over mean demand, times the probability that demand will exceed the supply.

Let us now see what Equation (6.8) has to tell us about the brooms with which we began this example. The average demand is $a = 10$ brooms per week, there are $k = 15$ brooms on hand each Monday morning. The lost sales expectation, therefore, is

$$\mu = 10P_{15}(1) - (15-10)\left[1 - \sum_{n=0}^{15} P_n(1)\right].$$

Since

$$P_{15}(1) = \frac{e^{-10}(10)^{15}}{15!} = 0.035,$$

from Equation (6.3), and

$$\sum_{n=0}^{15} P_n(1) = P_{15 \text{ or less}}(1) = 0.951,$$

from Example 4, the value of μ is

$$\mu = 10(0.035) - 5(1 - 0.951) = 0.1022.$$

Thus, on the average, we lose approximately one sale every 10 weeks. The loss represents approximately one percent of demand, since the total demand in 10 weeks would average 100, and we would lose one of these potential sales.

For $a = 10$, we obtain the average number μ of sales lost for various choices of k, the number of brooms stocked:

for $k = 16$, $\mu = 0.0532$;
for $k = 17$, $\mu = 0.0279$;
for $k = 18$, $\mu = 0.0156$;

(see Exercise 8). Thus, with $k = 18$, we expect an average of 1.5 sales lost every 100 weeks, which is around 3/10 of one percent of demand. This sales loss represents a substantial improvement over the result with $k = 15$, but the penalty is having to stock three more brooms.

Exercise 8. Obtain the values of μ for $k = 16, 17, 18$ in the preceding examples.

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7. CONCLUSION

We close with a few comments to indicate the introductory character of this unit. We have made no attempt to be complete, as you can see by asking some relatively easy questions about our examples. Our purpose will have been served if you have seen another way to use mathematics in practical, everyday situations. Perhaps some of you may also be inspired to learn more about the fascinating subject of queueing theory in the field of operations research.

8. MODEL EXAM

1. Suppose that at a certain hospital an average of 4 babies per day are born, and that each birth ties up a certain piece of delivery room equipment for two hours. The hospital has 3 of these pieces of equipment.
   a. Find the fraction of time during which all 3 pieces are in use;
   b. Convert the probability found in (a) to a rate in days of how often we expect the equipment to be inadequate.
2. Suppose there are 10 radar speed check points spaced randomly along a 2,000-mile stretch of interstate highway. Find the probability that a speeding motorist will not pass a radar site during a 300-mile drive on this highway.

9. ANSWERS TO EXERCISES

1. (a) $\mu = at = 6(1/2) = 3.$

   (b) $P_0(1/2) + P_1(1/2) + P_2(1/2) = e^{-3}(1 + 3 + 9/2) \approx 0.423.$

2. With $at = 3$, use the cumulative distribution function for the inter-arrival times:

   $F(1/2) = 1 - e^{-3} \approx 0.95$

3. Here $a = 10$, $b = 5$, $r = 2$, $c = 3$, $K = 12$.

   (a) Use Equation (4.6):

$$p_0 = \frac{1}{1 + 2 + 2 + \frac{8}{6}\ \frac{1 - (2/3)^{10}}{1 - (2/3)}}$$

$$= \frac{1}{5 + 4\ (1 - (2/3)^{10})} \approx 0.112.$$

   (b) Use Equation (4.9):

$$p_0 = \frac{1}{1 + 2 + 2 + \frac{8}{6}\ \frac{1}{1 - 2/3}} = \frac{1}{5 + 4} \approx 0.111.$$

   The results compare favorably, with relative error around 0.0009.

   (c) Use Equation (4.8):

   $p_1 = 2p_0, \quad p_2 = 2p_0.$

   The required probability is

   $p_0 + p_1 + p_2 = 5p_0 \approx 5(0.112) \approx 0.56.$

   (d) The given event is complementary to the event of part (c), so the required value is about $1 - 0.56 = 0.44$.

   (e) Use Equation (4.7):

$$p_{12} \approx \frac{2^{12}}{3!\ 3^9}\ (0.1119741346) \approx 0.00388.$$

   (f) The simplified approximation is

$$p_{12} \approx \frac{2^{12}}{3!\ 3^9}\ (1/9) \approx 0.00385.$$

The results agree to 4 decimal places.

4. For this case $c = 3$, but r remains $5/3$.

(a) $$p_0 = \frac{1}{1 + 5/3 + (5/3)^2/2 + ((5/3)^3/3)\,\frac{1}{1 - 5/9}} \approx 0.2087$$

$$p_0 = \frac{5}{3} p_0 \approx 0.3478$$

$$p_2 = \frac{5}{6} p_1 \approx 0.2899$$

$$p_3 = \frac{5}{9} p_2 \approx 0.1159$$

$$p_4 = \frac{5}{9} p_3 \approx 0.0644 .$$

(b) The probability that all 3 barbers will be busy equals the probability that at least 3 customers are in the system. Hence

$$\sum_{n=3}^{\infty} p_n = 1 - (p_0 + p_1 + p_2) \approx 0.1536.$$

Thus, only 15% of the arriving customers have any wait at all.

(c) The probability that 2 barbers will be idle is

$$p_0 + p_1 \approx 0.5565 > 1/2.$$

5. We have $c = 4$ and $r = 1/8$, so

(a) $$p_4 = \frac{(1/8)^4/24}{1 + 1/8 + (1/8)^2/2 + (1/8)^3/6 + (1/8)^4/24}$$

$$= 1/111,393.$$

(b) The loss rate is $3 \times 365 \times p_4 \approx 0.0098$ per year, or approximatel once every 101 years.

6. Here $c = 2$ and $r = 1/3$, so

(a) $$p_2 = \frac{(1/3)^2/2}{1 + 1/3 + (1/3)^2/2} = 1/25 = 0.04.$$

Thus, 4% of the calls are lost, and at 40 calls per day the loss rate is 1.6 per day, a ten-fold increase, but still relatively small.

(b) Using Equation (4.10), we obtain the value

$$\frac{((1/3)^2/2)\ \frac{1}{1 - 1/6}}{1 + 1/3 + ((1/3)^2/2)\ \frac{1}{1 - 16}} = \frac{1}{21} \approx 0.048.$$

This value is equivalent to a loss rate of approximately 1.9 calls per day.

7. $P_{17 \text{ or less}}(1) \approx 0.986$, for which the loss rate is one out of every 70 weeks. $P_{18 \text{ or less}}(1) \approx 0.993$, for which the loss rate is one out of every 139 weeks. Thus the required number of brooms is 18.

8. Answers are contained in Example 5.

10. ANSWERS TO MODEL EXAM

1. If we choose hours as the basic unit of time, then the average arrival rate of babies is every 6 hours, or 1/6 per hour.

   a. With $a = 1/6$, $b = 1/2$, we have $r = 1/3$, and Erlang's loss formula yields

$$p_3 = \frac{(1/3)^3/3!}{1 + 1/12 + (1/3)^2/2! + (1/3)^3/3!} = 1/226.$$

   b. Since 4 babies arrive per day on the average the result of part (a) means an inadequacy rate of once every 56.5 days

2. If we regard the radar sites as "arrivals", then 10 sites in 2,000 miles means an average of 1 site every 200 miles, so the average arrival rate a is 1/200. The probability that the motorist will not pass a radar site during a 300 mile stretch equals the probability that there will be no arrivals in an interval of length 300. The required value can be obtained from Equation (2.3) with $t = 300$ and $n = 0$: we have $at = 1.5$, and

$$P_0(300) = e^{-1.5}\frac{(1.5)^0}{0!} = e^{-1.5} \approx 0.22.$$

11. APPENDIX A: BASIC PROBABILITY CONCEPTS

In this unit we use relatively standard concepts of elementary probability theory. The basic notion is that of an "experiment", which is described by a triple (Ω, E, P), where:

- Ω is a set (the possible outcomes);
- E is a collection of subsets of Ω (the events);
- P is a "probability function".

The function P assigns probabilities to events according to the following rules:

(a) For every event A we have $0 \leq P(A) \leq 1$;

(b) $P(\Omega) = 1$;

(c) For every sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint events we have

$$P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } \ldots) = \sum_{n=1}^{\infty} P(A_n).$$

When E is a finite collection of sets, as is the case when Ω is a finite set, condition (c) may be replaced by:

(c)' For mutually exclusive events A and B we have

$$P(A \text{ or } B) = P(A) + P(B).$$

The following properties are fundamental and can be obtained easily from the defining relations above:

For any events A, B we have

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B); \tag{11.1}$$

For any event A we have

$$P(\text{not } A) = 1 - P(A). \tag{11.2}$$

We define two vital and closely related concepts. If B is an event with $P(B) > 0$, then the *conditional probability of an event* A, *given that* B *has occurred*, $P(A|B)$, is defined by:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}. \tag{11.3}$$

The related concept is statistical independence: events A and B are said to be *statistically independent* if

(11.4) $\quad P(A \text{ and } B) = P(A)\,P(B)$

Thus, if $P(A) > 0$ and $P(B) > 0$, then A and B are statistically independent if and only of $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

Conditional probability provides a most useful formula for finding probabilities. If events $A_1, \ldots, A_n$ are mutually exclusive and exhaustive, then for any event B we have

$$(11.5) \qquad P(B) = \sum_{i=1}^{n} P(B|A_i)\,P(A_i)$$

provided, of course, that each $P(A_i) > 0$.

If A consists of a single outcome, i.e. if $A = \{x\}$, then we denote the probability of A by P(x) instead of the formally correct $P(\{x\})$.

In many experiments the outcomes may be represented by real numbers, for example the numbers on the faces of a die, the number of dollars in the payoffs to players of a gambling game, and the number of customer arrivals at a store in an hour of the business day. When this happens, we many define the *mathematical expectation* of the experiment. If there are exactly n distinct possible real outcomes, then the mathematical expectation (or *mean*, or *average*) μ is defined by:

$$(11.6) \qquad \mu = \sum_{i=1}^{n} x_i P(x_i).$$

If the outcomes form an infinite sequence $\{x_n\}_{n=1}^{\infty}$, then the mean is

$$(11.6)' \qquad \mu = \sum_{n=1}^{\infty} x_n P(x_n).$$

In either case, μ is the average of the outcomes, weighted according to their likelihood of occurrence.

It may be that the possible outcomes of an experiment form an entire real interval rather than a discrete set. For example, consider the instant in time when the first customer of the day arrives at a store, or the distance between two cars in a traffic flow. In this case, probability is calculated from a *probability density function* f, which has the following properties:

(11.7) $\quad f(x) \geq 0$ for all x;

(11.8) $\quad \int_{-\infty}^{\infty} f(x)dx = 1.$

In this case the distribution is also said to be *continuous* and we calculate probabilities according to the following rule: If A is an interval of possible outcomes, say from a to b, then

(11.9) $\quad P(A) = \int_a^b f(x)dx.$

Thus, we may interpret the value of f(x) as follows: If B is an interval which contains x and which is of very small length Δx, then

(11.10) $\quad P(B) \approx f(x)\Delta x.$

The mean of a continuous distribution is given by

(11.11) $\quad \mu = \int_{-\infty}^{\infty} xf(x)dx,$

which is the analogue of Equation (11.6) and (11.6)'. For a continuous distribution the *cumulative distribution function* F is given by

(11.12) $\quad F(x) = \int_{-\infty}^{x} f(u)du.$

In this case we have the relation

(11.13) $\quad F'(x) = f(x)$

for all x, by the fundamental theorem of calculus.

12. APPENDIX B: DERIVATION OF ERLANG'S FORMULA

Here we derive Erlang's loss formula (Equation (4.4)), which is appropriate when no queue is allowed to form. This is the case in which the total system size K equals the number of servers c.

The basic assumptions for the arrivals are indicated in Section 2, and we shall denote the mean arrival rate by a here, too. One basic assumption on departures was described in Section 4. Namely, for any small interval from t to $t + \Delta t$ we have the conditional probability approximation

$$P(\text{service completion in } [t, t + \Delta t] \mid \text{service at } t) \approx b\Delta t$$

for each of the c servers. We also assume that for any small interval from t to $t + \Delta t$ the probability of more than one arrival or departure is of the order of Δt^2 and is therefore neglibible.

Consider a short interval of time from t to $t + \Delta t$. Let n be the number of customers in the system at time t. If $1 \leq n \leq c$, then at time $t + \Delta t$ there may be $n - 1$, n, or $n + 1$ customers, which corresponds to the respective possibilities of 1 departure and 0 arrivals in the interval, 0 departures and 0 arrivals, and 0 departures and 1 arrival. If $n = 0$, then we may have only 0 or 1 customer in the system at $t + \Delta t$; if $n = c$, we may have $n - 1$ or n. In addition, the conditional probability of 1 departure in the interval $[t, t + \Delta t]$, given n servers occupied at t is approximately $nb\Delta t$. We also assume statistical independence between arrivals and departures. Thus, the conditional probabilities for the number of customers in the system at $t + \Delta t$, given n in the system at t, are as follows for $1 \leq n \leq c - 1$:

$n + 1$	$a\Delta t(1 - nb\Delta t) \approx a\Delta t$

(1 arrival and no departures);

$\underline{n}$ $(1 - a\Delta t)(1 - nb\Delta t) \approx 1 - a\Delta t - nb\Delta t$
(no arrivals and no departures);

$\underline{n - 1}$ $(1 - a\Delta t)(nb\Delta t) \approx nb\Delta t$
(no arrivals and 1 departure).

When $n = 0$ at time t, the conditional probabilities for the number of customers in the system at time $t + \Delta t$ are

$\underline{1}$ $a\Delta t$
(1 arrival)

$\underline{0}$ $1 - a\Delta t$
(no arrivals).

When $n = c$ at time t the corresponding probabilities are

$\underline{c - 1}$ $nb\Delta t$
(1 departure)

$\underline{c}$ $1 - nb\Delta t$
(no departures).

Let p_n denote the probability that there will be n customers in the system at time t. As noted in Section 4, p_n really depends on t, but in many practical situations, the dependence on t becomes negligible as the system operation settles down to its steady state. Thus, the probability that there will be n customers in the system at time $t + \Delta t$ must be (approximately) p_n. But there is an alternate expression; for there may have been n customers at time t (with probability p_n) and no change during the interval (probability $1 - a\Delta t - nb\Delta t$); or there may have been $n - 1$ (probability p_{n-1}) and an increase of 1 (probability $a\Delta t$); or there may have been $n + 1$ (probability p_{n+1}) and a decrease of 1 (probability $(n + 1)b\Delta t$.

Thus, by Equation (11.5) we obtain

$$p_n = p_n(1 - a\Delta t - nb\Delta t) + p_{n-1}(a\Delta t) + p_{n+1}((n + 1)b\Delta t)$$

which simplifies to

$$(12.1) \qquad (a + nb)p_n = ap_{n-1} + b(n + 1)p_{n+1}.$$

(Note that the Δt drops out. Ultimately this is where our ignoring of "negligible" quantities is mathematically jusitified.)

Equation (12.1) is valid for $n = 1, 2, \ldots, c-1$, but not for $n = 0$ or $n = c$. The correct formulas for $n = 0$ and $n = c$ are:

$$p_0 = p_0(1 - a\Delta t) + p_1(b\Delta t),$$

or

$$(12.2) \qquad ap_0 = bp_1;$$

$$p_c = p_c(1 - cb\Delta t) + p_{c-1}(a\Delta t),$$

or

$$(12.3) \qquad bcp_c = ap_{c-1}.$$

Thus, we obtain a system of equations

$$(12.4) \qquad \begin{cases} ap_0 = bp_1 \\ (a + b)p_1 = ap_0 + 2bp_2 \\ (a + 2b)p_2 = ap_1 + 3bp_3 \\ \quad \vdots \\ (a + (c - 1)b)p_{c-1} = ap_{c-2} + cbp_c \\ bcp_c = ap_{c-1}. \end{cases}$$

We solve this system by first expressing the p_n's in terms of p_0 and then using the condition

$$(12.5) \qquad \sum_{n=0}^{c} p_n = 1$$

to find p_0. Add the equations for $n = 0$ and $n = 1$ and

cancel like terms to obtain

(12.6) $$ap_1 = 2bp_2.$$

Now add Equation (12.6) to the equation for n = 2 from the system (12.4) and simplify to obtain

$$ap_2 = 3bp_3.$$

Continuing in a similar way we find

$$ap_3 = 4bp_4$$

$$\vdots$$

$$ap_{c-1} = cbp_c.$$

(Note that the final equation is identical to the one obtained earlier for n = c.)

Now let a/b = r, the traffic intensity. Then the equation above may be written in the form

$$(12.7)\quad \begin{cases} p_1 = rp_0 \\ p_2 = (r^2/2)p_0 \\ p_3 = (r^3/3!)p_0 \\ \vdots \\ p_c = (r^c/c!)p_0. \end{cases}$$

Now substitute the results from (12.7) into Equation (12.5) to find p_0:

$$p_0(1 + r + r^2/2 + \ldots + r^c/c!) = 1,$$

from which

$$(12.8)\quad p_0 = \frac{1}{1 + r + r^2/2 + \ldots + r^c/c!}.$$

Erlang's first formula, Equation (4.1), now follows from (12.7) and (12.8):

$$(12.9) \qquad p_n = \frac{r^n/n!}{\sum_{i=0}^{c} r^i/i!}, \quad n = 0, 1, \ldots, c.$$

Erlang's loss formula now follows from Equation (12.9) for the value $n = c$.

Derivations of Equations (4.5) and (4.8) can be carried out in a similar way. We remark here only that the change in the formula at $n = c$ results from the fact that for $n > c$ the conditional probability of a service completion in the interval $[t, t + \Delta t]$, given n servers busy is no longer $nb\Delta t$, but instead is $cb\Delta t$.

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STUDENT FORM 2

Unit Questionnaire

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Name_____ Unit No._____ Date_____

Institution_____ Course No._____

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much Longer	Somewhat Longer	About the Same	Somewhat Shorter	Much Shorter
____	____	____	____	____

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain_____

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain_____

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)